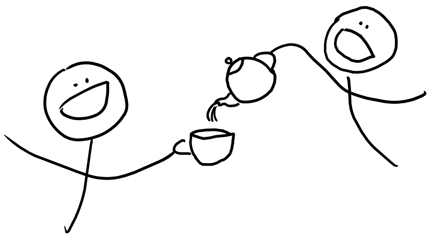
2021 Year 11 Specialist – Investigation 1

Take-home part – Syllabus Topic 1.1 (points 1.1.1-1.1.9)

Read the following and answer the questions in the text. Then complete the investigation questions at the end.

Ms Earl and Mr Grey are tea blenders. Together, they have come up with nine slightly different varieties for a new Orange Pekoe tea, and have reached the testing phase in the development process.



They recruit 10 members of the public to take part in the testing. Each person is given all nine tea varieties to taste, and then asked to rank them from 1 to 9 in order of preference.

After the first few trials, Ms Earl and Mr Grey notice a common complaint from the testers:

"Hmm," said Ms Earl, "Perhaps we should give each tester only a few tea varieties to try – maybe just three each?"

"OK," says Mr Grey, "But will we still be able to get meaningful results from our trials?"

"I don't know!" Ms Earl replies. "We should investigate!"

* * *

They decide to start their investigation by considering a simpler hypothetical situation, involving just five varieties of tea.

"Let's see what happens if we use just five testers," says Mr Grey. "I'll assign each tester (numbered 1 to 5) a combination of 3 varieties (labelled A, B, C, D and E) to taste and rank. I'll randomise the distribution by drawing the letters out of a hat."

Tester	1	2	3	4	5
Tea varieties	BDC	CEA	ACE	EBC	DBC

"That looks ok," says Ms Earl, "Each tea variety gets tested by at least one person. But… oh no! I've noticed a problem! Varieties A and B are never assigned to the same person. That means that A and B are never compared and ranked under identical tasting conditions!"

"Hm. Maybe we need to be more systematic to ensure that every pair of varieties gets tried together by at least one person. Perhaps we could assign every possible combination of three tea varieties to a different tester?"

1. Show with a calculation that the number of testers needed for this is 10.

Ms Earl comes up with the following distribution of varieties among testers.

Tester	1	2	3	4	5	6	7	8	9	10
Variety	ABC	BCD	CDE	DEA	EAB	ABD	BCE	CDA	DEB	EAC

"Much better!" says Mr Grey. "Now each pair of tea varieties is compared by at least one tester. But… 10 testers *does* seem like a lot – and costly – for just five varieties. After all, varieties A and B get tested together by Tester 1, Tester 5 *and* Tester 6, when just one tester comparing A and B might be enough for our first phase of trials."

2. For each possible pair of tea varieties from A, B, C, D and E, determine the number of testers in the distribution above who try that pair together.

"I see what you mean," says Ms Earl. "You know what would be ideal? A distribution where every possible pair of teas is tried together by *exactly one* tester."

"That would indeed be very efficient!" says Mr Grey, "In fact, it's such a good idea I think we should give it a name!"

<u>Definition</u>: A distribution of n tea varieties among m testers is called an *ideal 3-distribution* if:

- each of the m testers is assigned a combination of 3 tea varieties (called a 3-set) to try; and
- each pair of the n teas is tried together by *exactly one* tester.

After a while, Ms Earl says: "Oh no! I think I've just discovered that it's impossible to find an ideal 3-distribution when $n = 5$."

3. Suppose that $n = 5$.
   a) Calculate the number of pairs of tea varieties that can be made from these 5.
   b) Calculate the number of pairs that occur in a single 3-set.
   c) Hence, explain why an ideal 3-distribution cannot exist for $n = 5$.

A while later, Mr Grey says: "Oh dear! I think we can also prove that it's impossible to find an ideal 3-distribution if n is *any even number*!"

4. Consider n tea varieties labelled A, B, C etc. and suppose that there exists an ideal 3-distribution for these n varieties.
   a) Consider two different 3-sets in the distribution which both contain the letter A. How many letters must these sets have in common? Why?
   b) Is the letter B guaranteed to be in one of the 3-sets containing A? How do you know? Is the same true for any other letter?
   c) What do your answers to parts (a) and (b) imply about the value of $n - 1$?
   d) Hence, explain why there cannot exist an ideal 3-distribution if n is an even number.

"That's disappointing!" says Mr Grey. "But we should still investigate whether ideal 3-distributions could exist for other values of n."

"Certainly," says Ms Earl. "I'm beginning to enjoy this!"

Now investigate the following questions:

1. *For which odd values of n could there possibly exist an ideal 3-distribution of n tea varieties among some number m of testers?*

2. *If such a distribution could exist, what methods could be used to find it?*

To investigate the second question, consider only values of n less than or equal to 11. (You are welcome to try with larger values too, but these may be much more difficult!)